On Montel and Montel-Popoviciu theorems in several variables

A. G. Aksoy and J. M. Almira

Abstract. We present an elementary proof of a general version of Montel's theorem in several variables which is based on the use of tensor product polynomial interpolation. We also prove a Montel-Popoviciu's type theorem for functions $f : \mathbb{R}^d \to \mathbb{R}$ for $d > 1$. Furthermore, our proof of this result is also valid for the case $d = 1$, differing in several points from Popoviciu's original proof. Finally, we demonstrate that our results are optimal.

1. Introduction

The study of functional equations has substantially grown in the last three decades [1], [10], [22], [23], [34], [37]. Research in this area is provoking interesting questions concerning characterizations of polynomials [4], [30] and exponential polynomials [42], [45]. These concrete questions have close connections to spectral analysis and synthesis and have found its way to a number of interesting applications [42], [43], [44]. Both Montel and Popoviciu, in their seminal papers [32], [33] and [36], used Fréchet functional equation, with some additional regularity conditions, for the characterization of polynomials. In this paper, we are interested in generalizing these results, by using some new tools, in the several variable setting. Our aim is to show that under suitable conditions we again end up with polynomials.

Concretely, we are interested in a special regularity result for the the functional equation $\Delta_h^{m+1} f(x) = 0$, where $f : \mathbb{R}^d \to \mathbb{R}$ and the higher differences operator Δ_h^{m+1} is inductively defined by $\Delta_h^1 f(x) = f(x+h) - f(x)$, and $\Delta_h^{n+1} f(x) = \Delta_h(\Delta_h^n f)(x)$, $n = 1, 2, \cdots$. This equation was introduced in

the literature for functions $f : \mathbb{R} \to \mathbb{R}$, by M. Fréchet in 1909 as a particular case of the functional equation

$$\Delta_{h_1 h_2 \cdots h_{m+1}} f(x) = 0 \quad (x, h_1, h_2, \ldots, h_{m+1} \in \mathbb{R}), \qquad (1.1)$$

where $f : \mathbb{R} \to \mathbb{R}$ and $\Delta_{h_1 h_2 \cdots h_s} f(x) = \Delta_{h_1} \left(\Delta_{h_2 \cdots h_s} f \right)(x)$, $s = 2, 3, \cdots$. In particular, after Fréchet's seminal paper [15], the solutions of (1.1) are named "polynomial functions" by the functional equations community, since it is known that, under very mild regularity conditions on f, if $f : \mathbb{R} \to \mathbb{R}$ satisfies (1.1), then $f(x) = a_0 + a_1 x + \cdots a_m x^m$ for all $x \in \mathbb{R}$ and certain constants $a_i \in \mathbb{R}$. For example, in order to have this property, it is enough for f being bounded on a set $A \subseteq \mathbb{R}$ of positive Lebesgue measure $|A| > 0$ (see, for example, [42] for a proof of this result). Equation (1.1) can also be studied for functions $f : X \to Y$ whenever X, Y are two $\mathbb{Q}$-vector spaces and the variables $x, h_1, \cdots, h_{m+1}$ are assumed to be elements of X:

$$\Delta_{h_1 h_2 \cdots h_{m+1}} f(x) = 0 \quad (x, h_1, h_2, \ldots, h_{m+1} \in X). \qquad (1.2)$$

In this context, the general solutions of (1.2) are characterized as functions of the form $f(x) = A_0 + A_1(x) + \cdots + A_m(x)$, where A_0 is a constant and $A_k(x) = A^k(x, x, \cdots, x)$ for a certain k-additive symmetric function $A^k : X^k \to Y$ (we say that A_k is the diagonalization of A^k). In particular, if $x \in X$ and $r \in \mathbb{Q}$, then $f(rx) = A_0 + r A_1(x) + \cdots + r^m A_m(x)$. Furthermore, it is known that $f : X \to Y$ satisfies (1.2) if and only if it satisfies

$$\Delta_h^{m+1} f(x) := \sum_{k=0}^{m+1} \binom{m+1}{k} (-1)^{m+1-k} f(x + kh) = 0 \quad (x, h \in X). \qquad (1.3)$$

A proof of this fact follows directly from Djoković's Theorem [13] (see also [20, Theorem 7.5, page 160], [26, Theorem 15.1.2., page 418] and, for a completely different new proof, [45]).

In 1935 P. Montel [32] studied Fréchet's functional equation from a fresh perspective (see also [33]). Indeed, he was not motivated by Fréchet's paper but by a much older one by Jacobi [21], who in 1834 proved that if $f : \mathbb{C} \to \widehat{\mathbb{C}}$ is a non-constant meromorphic function defined on the complex numbers, then $\mathfrak{P}_0(f) = \{ w \in \mathbb{C} : f(z + w) = f(z) \text{ for all } z \in \mathbb{C} \}$, the set of periods of f, is a discrete subgroup of $(\mathbb{C}, +)$. This reduces the possibilities to the following three cases: $\mathfrak{P}_0(f) = \{0\}$, or $\mathfrak{P}_0(f) = \{ n w_1 : n \in \mathbb{Z} \}$ for a certain complex number $w_1 \neq 0$, or $\mathfrak{P}_0(f) = \{ n_1 w_1 + n_2 w_2 : (n_1, n_2) \in \mathbb{Z}^2 \}$ for certain complex numbers w_1, w_2 satisfying $w_1, w_2 \neq 0$ and $w_1 / w_2 \notin \mathbb{R}$. In particular, these functions cannot have three independent periods and there exist meromorphic functions $f : \mathbb{C} \to \widehat{\mathbb{C}}$ with two independent periods w_1, w_2 as soon as $w_1 / w_2 \notin \mathbb{R}$. These functions are called doubly periodic (or elliptic) and have an important role in complex function theory [24]. Analogously, if the function $f : \mathbb{R} \to \mathbb{R}$ is continuous and non-constant, it does not admit two $\mathbb{Q}$-linearly independent periods. Obviously, Jacobi's theorem can be formulated as a result which characterizes the constant functions as those meromorphic functions $f : \mathbb{C} \to \widehat{\mathbb{C}}$ which solve a system of functional

equations of the form
$$\Delta_{h_1} f(z) = \Delta_{h_2} f(z) = \Delta_{h_3} f(z) = 0 \quad (z \in \mathbb{C}), \tag{1.4}$$
for three independent periods $\{h_1, h_2, h_3\}$ (i.e., $h_1\mathbb{Z} + h_2\mathbb{Z} + h_3\mathbb{Z}$ is a dense subset of $\mathbb{C}$). For the real case, the result states that, if $h_1, h_2 \in \mathbb{R}\backslash\{0\}$ are two nonzero real numbers and $h_1/h_2 \notin \mathbb{Q}$, the continuous function $f : \mathbb{R} \to \mathbb{R}$ is a constant function if and only if it solves the system of functional equations
$$\Delta_{h_1} f(x) = \Delta_{h_2} f(x) = 0 \ (x \in \mathbb{R}). \tag{1.5}$$
In [32], [33] Montel substituted Δ_h^{m+1} for Δ_h in the equations $(1.4), (1.5)$ and proved that these equations are appropriate for the characterization of ordinary polynomials.

Theorem 1.1 (Montel). *Assume that $f : \mathbb{C} \to \mathbb{C}$ is an analytic function which solves a system of functional equations of the form*
$$\Delta_{h_1}^{m+1} f(z) = \Delta_{h_2}^{m+1} f(z) = \Delta_{h_3}^{m+1} f(z) = 0 \quad (z \in \mathbb{C}) \tag{1.6}$$
for three independent periods $\{h_1, h_2, h_3\}$. Then $f(z) = a_0 + a_1 z + \cdots + a_m z^m$ is an ordinary polynomial with complex coefficients and degree $\leq m$. Furthermore, if $\{h_1, h_2\} \subset \mathbb{R}\backslash\{0\}$ satisfy $h_1/h_2 \notin \mathbb{Q}$, the continuous function $f : \mathbb{R} \to \mathbb{R}$ is an ordinary polynomial with real coefficients and degree $\leq m$ if and only if it solves the system of functional equations
$$\Delta_{h_1}^{m+1} f(x) = \Delta_{h_2}^{m+1} f(x) = 0 \ (x \in \mathbb{R}). \tag{1.7}$$

To highlight the relationship between his theorem and Jacobi's results, Montel named "generalized periods" of order $m+1$ of f to the vectors h such that $\Delta_h^{m+1} f = 0$.

Montel's result uses the regularity properties of Fréchet's functional equation in a new non-standard form. Indeed, the idea is now not to conclude the regularity of f from the assumption that it solves the equation for all h and it satisfies some mild regularity condition, but to assume that f is globally regular (indeed, it is continuous everywhere) and to describe the minimal set Γ of generalized periods h such that, if the equation is solved for all $h \in \Gamma$, then it is also solved for all h. Thus, the regularity of the solution is assumed and used to conclude that, in order to be a solution of Fréchet's equation it is enough to have a very small set of generalized periods.

In his paper [33], Montel also studied the equation (1.3) for $X = \mathbb{R}^d$, with $d > 1$, and $f : \mathbb{R}^d \to \mathbb{C}$ continuous, and for $X = \mathbb{C}^d$ and $f : \mathbb{C}^d \to \mathbb{C}$ analytic. Concretely, he stated (and gave a proof for $d = 2$) the following result.

Theorem 1.2 (Montel's Theorem in several variables). *Let $\{h_1, \cdots, h_s\} \subset \mathbb{R}^d$ be such that*
$$h_1\mathbb{Z} + h_2\mathbb{Z} + \cdots + h_s\mathbb{Z} \text{ is a dense subset of } \mathbb{R}^d, \tag{1.8}$$
and let $f \in C(\mathbb{R}^d, \mathbb{C})$ be such that $\Delta_{h_k}^m (f) = 0$, $k = 1, \cdots, s$. Then $f(x) = \sum_{|\alpha| < N} a_\alpha x^\alpha$ for some $N \in \mathbb{N}$, some complex numbers a_α, and all $x \in \mathbb{R}^d$. Thus, f is an ordinary complex-valued polynomial in d real variables.

Consequently, if $d = 2k$, $\{h_i\}_{i=1}^s$ satisfies (1.8), the function $f : \mathbb{C}^k \to \mathbb{C}$ is holomorphic and $\Delta_{h_k}^m(f) = 0$, $k = 1, \cdots, s$, then $f(z) = \sum_{|\alpha|<N} a_\alpha z^\alpha$ is an ordinary complex-valued polynomial in k complex variables.

The finitely generated subgroups of $(\mathbb{R}^d, +)$ which are dense in $\mathbb{R}^d$ have been actively studied and, in fact, they can be characterized in several ways. For example, in [48, Proposition 4.3], the following theorem is proved:

Theorem 1.3. *Let $G = h_1\mathbb{Z} + h_2\mathbb{Z} + \cdots + h_s\mathbb{Z}$ be the additive subgroup of $\mathbb{R}^d$ generated by the vectors $\{h_1, \cdots, h_s\}$. The following statements are equivalent:*

(i) *G is a dense subgroup of $\mathbb{R}^d$.*
(ii) *If $h_k = (a_{1k}, a_{2k}, \cdots, a_{dk})$ are the coordinates of h_k with respect to the canonical basis of $\mathbb{R}^d$ ($k = 1, \cdots, s$), then the matrices*

$$A(n_1, \cdots, n_s) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d1} & a_{d2} & \cdots & a_{ds} \\ n_1 & n_2 & \cdots & n_s \end{bmatrix}.$$

have rank equal to $d + 1$, for all $(n_1, \cdots, n_s) \in \mathbb{Z}^s \setminus \{(0, \cdots, 0)\}$.

A simple case which has motivated the study of dense subgroups of $(\mathbb{R}^d, +)$, is the following one:

Corollary 1.4 (Kronecker's theorem). *Given $\theta_1, \theta_2, \cdots, \theta_d \in \mathbb{R}$, the group $\mathbb{Z}^d + (\theta_1, \theta_2, \cdots, \theta_d)\mathbb{Z}$ (which is generated by exactly $d + 1$ elements) is a dense subgroup of $\mathbb{R}^d$ if and only if*

$$n_1\theta_1 + \cdots + n_d\theta_d \notin \mathbb{Z}, \text{ for all } (n_1, \cdots, n_d) \in \mathbb{Z}^d \setminus \{(0, \cdots, 0)\}$$

in other words, this group is dense in $\mathbb{R}^d$ if and only if $\{1, \theta_1, \cdots, \theta_d\}$ forms a linearly independent system over $\mathbb{Q}$.

Proof. The vectors $\{e_k\}_{k=1}^d \cup \{(\theta_1, \cdots, \theta_d)\}$, where

$$e_k = (0, 0, \cdots, 1^{(k\text{-th position})}, 0, \cdots, 0), \quad k = 1, \cdots, d,$$

generate the group $G = \mathbb{Z}^d + (\theta_1, \theta_2, \cdots, \theta_d)\mathbb{Z}$. Thus, part (ii) from Theorem 1.3 guarantees that G is dense in $\mathbb{R}^d$ if and only if

$$
\det \begin{bmatrix}
\theta_1 & 1 & 0 & \cdots & 0 \\
\theta_2 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & & \ddots & \vdots \\
\theta_d & 0 & 0 & \cdots & 1 \\
n_0 & n_1 & n_2 & \cdots & n_d
\end{bmatrix}
$$

$$
= (-1)^{d+2}n_0 + \sum_{k=1}^{d}(-1)^{d+2+k}n_k(-1)^{k+1}\theta_k \det(I_{d-1})
$$

$$
= (-1)^d\left(n_0 - \sum_{k=1}^{d}n_k\theta_k\right) \neq 0
$$

for all $(n_0, n_1, \cdots, n_d) \in \mathbb{Z}^{d+1} \setminus \{(0, \cdots, 0)\}$, which is equivalent to claim that

$$
n_1\theta_1 + \cdots + n_d\theta_d \notin \mathbb{Z}, \text{ for all } (n_1, \cdots, n_d) \in \mathbb{Z}^d \setminus \{(0, \cdots, 0)\},
$$

which is what we looked for. To prove the last claim in the theorem, it is enough to observe that, if $\{1, \theta_1, \cdots, \theta_d\}$ forms a $\mathbb{Q}$-linearly dependent system, then there are rational numbers $r_i = n_i/m_i$, $i = 0, 1, \cdots, d$ (not all of them equal to zero), such that

$$
r_0 + r_1\theta_1 + \cdots + r_d\theta_d = 0,
$$

so that, multiplying both sides of the equation by $m = \prod_{k=0}^d m_k$ we get

$$
n_0^* + n_1^*\theta_1 + \cdots + n_d^*\theta_d = 0
$$

for certain natural numbers $n_0^*, n_1^*, \cdots, n_d^*$ (not all equal to zero). In particular,

$$
n_1^*\theta_1 + \cdots + n_d^*\theta_d \in \mathbb{Z}.
$$

$\square$

Montel's theorem was improved by his student T. Popoviciu [36] for the case $d = 1$ to the following:

Theorem 1.5 (Popoviciu, 1935). *Let $f : \mathbb{R} \to \mathbb{R}$ be such that*

$$
\Delta_{h_1}^{m+1}f(x) = \Delta_{h_2}^{m+1}f(x) = 0, \ (x \in \mathbb{R}),
$$

If $h_1/h_2 \notin \mathbb{Q}$ and f is continuous in at least $m + 1$ distinct points, then $f \in \Pi_m$.

In section 2 we prove, by very elementary means, a theorem which generalizes Montel's Theorem in several variables. In section 3 we prove a version of Montel-Popoviciu's theorem for functions $f : \mathbb{R}^d \to \mathbb{R}$ for $d > 1$. Furthermore, our proof is also valid for the case $d = 1$, and, in that case, it differs in several points from Popoviciu's original proof. In section 4 we prove that Popoviciu's original result is optimal, since, if $h_1, h_2 \in \mathbb{R}$ and $h_1/h_2 \notin \mathbb{Q}$, there exists $f : \mathbb{R} \to \mathbb{R}$ continuous in m distinct points which is not an ordinary polynomial and solves the system of equations $\Delta_{h_1}^{m+1}f(x) = \Delta_{h_2}^{m+1}f(x) = 0$.

Finally, in that section we also consider the optimality of Montel-Popoviciu's theorem in the several variables setting.

In this paper we use the following standard notation: Π_m denotes the space of complex polynomials of a real variable, with degree $\leq m$. $\Pi_{m,\max}^d$ denotes the space of complex polynomials of d real variables, with degree $\leq m$ with respect to each one of these variables. More precisely, $P(x_1, \cdots, x_d) \in \Pi_{m,\max}^d$ if and only if

$$P(x_1, \cdots, x_d) = \sum_{i_1=0}^{m} \sum_{i_2=0}^{m} \cdots \sum_{i_d=0}^{m} a_{i_1,\cdots,i_d} x_1^{i_1} \cdots x_d^{i_d},$$

with $a_{i_1,\cdots,i_d}$ being complex numbers. Finally, $\Pi_{m,\mathrm{tot}}^d$ denotes the space of complex polynomials of d real variables, with total degree $\leq m$; that is, $P(x_1, \cdots, x_d) \in \Pi_{m,\mathrm{tot}}^d$ if and only if

$$P(x_1, \cdots, x_d) = \sum_{i_1,\cdots,i_d \in \mathbb{N} \text{ and } i_1+\cdots+i_d \leq m} a_{i_1,\cdots,i_d} x_1^{i_1} \cdots x_d^{i_d},$$

with $a_{i_1,\cdots,i_d}$ being complex numbers. Of course, $\Pi_m = \Pi_{m,\max}^1 = \Pi_{m,\mathrm{tot}}^1$.

2. A generalization of Montel theorem

For the statement of the results in this section, we need to recall the following concept from interpolation theory: Given $\Lambda \subseteq \Pi^d$ a subspace of the space of polynomials in d real variables, and given $W \subseteq \mathbb{R}^d$, we say that W is a correct interpolation set (sometimes also referred as insolvent) for Λ if and only if for any function $f : W \to \mathbb{C}$ there exists a unique polynomial $P \in \Lambda$ such that $P(x) = f(x)$ for all $x \in W$. In particular, if W is a correct interpolation set for Λ and $P \in \Lambda$ satisfies $P_{|W} = 0$, then $P = 0$. These sets have been characterized for several spaces of polynomials Λ [46]. In particular, the technique of tensor product interpolation guarantees that the sets W of the form

$$W = \{x_0^1, x_1^1, \cdots, x_m^1\} \times \{x_0^2, x_1^2, \cdots, x_m^2\} \times \cdots \times \{x_0^d, x_1^d, \cdots, x_m^d\},$$

which form a rectangular grid of $(m+1)^d$ points in $\mathbb{R}^d$, are correct interpolation sets for $\Lambda = \Pi_{m,\max}^d$ (see, e.g., [19, page 295]).

Let G be an Abelian group, let $f : G \to \mathbb{C}$ be an arbitrary function. The functions $\Delta_h^n f : G \to \mathbb{C}$ are well defined and f is named a complex polynomial function of degree $\leq m$ on G if $\Delta_h^{m+1} f(x) = 0$ for all $x, h \in G$. If there exist additive functions $a_k : G \to \mathbb{C}$, $k = 1, \cdots, t$ and an ordinary complex polynomial $P \in \mathbb{C}[x_1, \cdots, x_t]$ of total degree $\leq m$ such that $f(x) = P(a_1(x), \cdots, a_t(x))$ for all $x \in G$, we just say that f is a polynomial on G. It is known that, for finitely generated Abelian groups, every polynomial function is a polynomial [42].

Given $\gamma = \{h_1, \cdots, h_s\} \subseteq G$, $a \in G$, and given $f : G \to \mathbb{C}$ a polynomial function of degree $\leq m$, there exists a unique polynomial $P = P_{a,\gamma} \in \Pi_{m,\max}^s$

such that

$$P(i_1, i_2, \cdots, i_s) = f_{i_1, \cdots, i_s} := f(a + \sum_{k=1}^{s} i_k h_k),$$

for all $0 \leq i_k \leq m$, $1 \leq k \leq s$, since $W = \{(i_1, \cdots, i_s) : 0 \leq i_k \leq m,\ k = 1, \cdots, s\}$ is a correct interpolation set for $\Pi_{m,\max}^s$. In all what follows, we denote this polynomial by $P_{a,\gamma}$.

Lemma 2.1. *Let G be a commutative group and $f : G \to \mathbb{C}$ be a function. If $\Delta_{h_k}^{m+1} f(x) = 0$ for all $x \in G$ and $k = 1, \cdots, s$, then*

$$P_{a,\gamma}(i_1, i_2, \cdots, i_s) = f(a + \sum_{k=1}^{s} i_k h_k), \ \text{for all } (i_1, \cdots, i_s) \in \mathbb{Z}^{d+1}.$$

Proof. Let us fix the values of $k \in \{1, \cdots, s\}$ and $i_1, \cdots, i_{k-1}, i_{k+1}, \cdots, i_s \in \{0, 1, \cdots, m\}$, and let us consider the polynomial of one variable

$$q_k(x) = P_{a,\gamma}(i_1, \cdots, i_{k-1}, x, i_{k+1}, \cdots, i_{d+1}).$$

Obviously $q_k \in \Pi_m^1$, so that

$$0 = \Delta_1^{m+1} q_k(0) = \sum_{r=0}^{m+1} \binom{m+1}{r} (-1)^{m+1-r} q_k(r)$$

$$= \sum_{r=0}^{m} \binom{m+1}{r} (-1)^{m+1-r} P_{a,\gamma}(i_1, \cdots, i_{k-1}, r, i_{k+1}, \cdots, i_{d+1}) + q_k(m+1)$$

$$= \sum_{r=0}^{m} \binom{m+1}{r} (-1)^{m+1-r} f(a + \sum_{(0 \leq j \leq s;\ j \neq k)} i_j h_j + r h_k) + q_k(m+1)$$

$$= \Delta_{h_k}^{m+1} f(a + \sum_{(0 \leq j \leq s;\ j \neq k)} i_j h_j) - f(a + \sum_{(0 \leq j \leq s;\ j \neq k)} i_j h_j + (m+1) h_k)$$

$$+ q_k(m+1)$$

$$= q_k(m+1) - f(a + \sum_{(0 \leq j \leq s;\ j \neq k)} i_j h_j + (m+1) h_k).$$

It follows that

$$\begin{aligned} q_k(m+1) &= P_{a,\gamma}(i_1, \cdots, i_{k-1}, (m+1), i_{k+1}, \cdots, i_s) \\ &= f(a + \textstyle\sum_{(0 \leq j \leq s;\ j \neq k)} i_j h_j + (m+1) h_k). \end{aligned} \qquad (2.1)$$

Let us now consider the unique polynomial $P \in \Pi_{m,max}^s$ which satisfies the Lagrange interpolation conditions

$$P(i_1, i_2, \cdots, i_s h_s) = f(a + \sum_{k=1}^{s} i_k h_k)$$

for all $0 \leq i_j \leq m$, $1 \leq j \leq s$, $j \neq k$, and all $1 \leq i_k \leq m+1$. We have already demonstrated, with formula (2.1), that this polynomial coincides with $P_{a,\gamma}$.

Furthermore, the very same arguments used to prove (2.1), applied to the polynomial $P = P_{a,\gamma}$, lead us to the conclusion that

$$P_{a,\gamma}(i_1, \cdots, i_{k-1}, (m+2), i_{k+1}, \cdots, i_s h_s)$$
$$= f(a + \sum_{(0 \leq j \leq s; \ j \neq k)} i_j h_j + (m+2)h_k)$$

In an analogous way, extracting this time the first term of the sum, and taking as starting point the equality

$$\Delta_{h_k}^{m+1} f(a + \sum_{(0 \leq j \leq s; \ j \neq k)} i_j h_j - h_k) = 0,$$

we conclude that

$$P_{a,\gamma}(i_1, \cdots, i_{k-1}, -1, i_{k+1}, \cdots, i_s)$$
$$= f(a + \sum_{(0 \leq j \leq s; \ j \neq k)} i_j h_j - h_k).$$

Repeating these arguments forward and backward infinitely many times, and for each $k \in \{1, \cdots, s\}$, we get

$$P_{a,\gamma}(i_1, i_2, \cdots, i_s)$$
$$= f(a + \sum_{k=1}^{s} i_k h_k), \text{ for all } (i_1, \cdots, i_s) \in \mathbb{Z}^{d+1},$$

which is what we wanted to prove. $\qquad\square$

Corollary 2.2. *Let G be a topological Abelian group. If $f : G \to \mathbb{C}$ satisfies $\Delta_{h_k}^{m+1} f(x) = 0$ for all $x \in G$ and $k = 1, \cdots, s$, then*

$$\Delta_h^{sm+1} f(x) = 0$$

for all $h \in H = h_1 \mathbb{Z} + \cdots + h_s \mathbb{Z}$.

Proof. Let $P_{a,\gamma}$ be the polynomial constructed in Lemma 2.1, and let $x \in \mathbb{R}^d$. Then

$$P_{x,\gamma}(i_1, i_2, \cdots, i_s) = f(x + \sum_{k=1}^{s} i_k h_k), \text{ for all } (i_1, \cdots, i_s) \in \mathbb{Z}^{d+1}.$$

Hence, if $h = \sum_{k=1}^{s} i_k h_k \in G$,

$$
\begin{aligned}
\Delta_h^{sm+1} f(x) &= \sum_{r=0}^{sm+1} \binom{sm+1}{r} (-1)^{sm+1-r} f(x + r \sum_{k=1}^{s} i_k h_k) \\
&= \sum_{r=0}^{sm+1} \binom{sm+1}{r} (-1)^{sm+1-r} P_{x,\gamma}(r(i_1, \cdots, i_s)) \\
&= \Delta_{(i_1, \cdots, i_s)}^{sm+1} P_{x,\gamma}(0) = 0,
\end{aligned}
$$

since $P_{x,\gamma}$ is a polynomial in s variables of total degree $\leq sm$. This proves the first part of the corollary. $\qquad\square$

Corollary 2.3. *If G is a topological Abelian group, $f : G \to \mathbb{R}$ is continuous and satisfies $\Delta_{h_k}^{m+1} f(x) = 0$ for all $x \in G$ and $k = 1, \cdots, s$ and $H = h_1\mathbb{Z} + \cdots + h_s\mathbb{Z}$ is a dense subgroup of G, then*

$$\Delta_h^{sm+1} f(x) = 0$$

for all $x, h \in G$. In other words, f is a continuous polynomial on G of degree $\leq sm$.

In particular, if $G = \mathbb{R}^d$, then $f \in \Pi_{sm,tot}^d$.

Proof. Let us now assume also that f is continuous and G is dense in $\mathbb{R}^d$. Then it is clear that $\Delta_h^{sm+1} f(x) = 0$ for all x, h in $\mathbb{R}^d$, and, if $G = \mathbb{R}^d$, Fréchet's Theorem implies that f is an ordinary polynomial. In fact, it is not difficult to prove that, in this case, f must have total degree $\leq sm$ (see, for example, [5, Theorem 3.1]). □

Remark 2.4. *Corollary 2.3 is optimal for $G = \mathbb{R}^d$ since, for each $s \geq d+1$ there exists $f : \mathbb{R}^d \to \mathbb{R}$ and $\{h_1, \cdots, h_s\} \subset \mathbb{R}^d$ such that $H = h_1\mathbb{Z} + \cdots + h_s\mathbb{Z}$ is dense in $\mathbb{R}^d$, $\Delta_{h_k}^{m+1} f(x) = 0$ for all $x \in \mathbb{R}^d$ and $k = 1, \cdots, s$, and $\Delta_h^{sm} f \neq 0$ for some $h \in H$. To prove this, let $\gamma = \{h_1, \cdots, h_s\} \subseteq \mathbb{R}^d$ be such that $G = h_1\mathbb{Z} + \cdots + h_s\mathbb{Z}$ is dense in $\mathbb{R}^d$, and let us assume that*

$$A_\gamma = \mathbf{col}[h_1, \cdots, h_s] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \ddots & \cdots & \vdots \\ a_{d1} & a_{d2} & \cdots & a_{ds} \end{bmatrix}$$

contains a set $\{a_{k1}, a_{k2}, \cdots, a_{ks}\}$ which forms a $\mathbb{Q}$-linearly independent set of real numbers. If we consider two vectors $(i_1, \cdots, i_s), (j_1, \cdots, j_s) \in \mathbb{Z}^s$, then

$$i_1 h_1 + \cdots + i_s h_s = j_1 h_1 + \cdots + j_s h_s \text{ if and only if } i_k = j_k \text{ for all } k = 1, \cdots, s.$$

Hence the function $f : \mathbb{R}^d \to \mathbb{R}$ given by

$$f(x) = \begin{cases} P(i_1, \cdots, i_s) & \text{if } x = i_1 h_1 + \cdots + i_s h_s \text{ and } (i_1, \cdots, i_s) \in \mathbb{Z}^s \\ 0 & \text{otherwise} \end{cases},$$

is well defined for any map $P : \mathbb{Z}^s \to \mathbb{R}$. Let $P \in \Pi_{m,\max}^s \subseteq \Pi_{sm,tot}^s$ be the polynomial $P(x_1, \cdots, x_s) = x_1^m x_2^m \cdots x_s^m$. Then

$$\Delta_{h_k}^{m+1} f(x) = 0, \quad \text{for all } x \in \mathbb{R} \text{ and } k = 1, \cdots, s,$$

and, on the other hand, if we consider the monomial $\varphi_N : \mathbb{R} \to \mathbb{R}$ defined by $\varphi_N(t) = t^N$, then

$$
\begin{aligned}
\Delta_{h_1 + \cdots + h_s}^{sm} f(0) &= \Delta_{(1,1,1,\cdots,1)}^{sm} P(0,0,\cdots,0) \\
&= \sum_{k=0}^{sm} \binom{sm}{k} (-1)^{sm-k} P(k, k, \cdots, k) \\
&= \sum_{k=0}^{sm} \binom{sm}{k} (-1)^{sm-k} k^{sm} \\
&= \Delta_1^{sm} \varphi_{sm}(0) = (sm)! \varphi_{sm}(1) = (sm)! \neq 0,
\end{aligned}
$$

*since the monomials φ_N satisfy the functional equation $\frac{1}{N!}\Delta_h^N \varphi_N(x) = \varphi_N(h)$.
Hence f satisfies our requirements.*

*To support our argument let us show an example of matrix A_γ satisfying
our hypotheses. Let $\{1, \theta_1, \cdots, \theta_d\}$ be a $\mathbb{Q}$-linearly independent set of real
numbers of size $d+1$ with $s = d+1$. Indeed, we impose $\theta_k = \pi^k$ for all k.
Let us consider the matrix*

$$A_\gamma = \mathbf{col}[h_1, \cdots, h_s] := \begin{bmatrix} \pi & 1 & \pi^2 & \cdots & \pi^d \\ \pi^2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ \pi^d & 0 & 0 & \cdots & 1 \end{bmatrix}$$

*Then part (ii) of Theorem 1.3 claims that $G = h_1\mathbb{Z} + \cdots + h_s\mathbb{Z}$ is dense in
$\mathbb{R}^d$ if and only if, for each $(n_0, \cdots, n_d) \in \mathbb{Z}^{d+1} \setminus \{(0, \cdots, 0)\}$,*

$$0 \neq \det(B_\gamma(n_0, \cdots, n_d)) := \det \begin{bmatrix} \pi & 1 & \pi^2 & \cdots & \pi^d \\ \pi^2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ \pi^d & 0 & 0 & \cdots & 1 \\ n_0 & n_1 & n_2 & \cdots & n_d \end{bmatrix}$$

*Now, a simple computation (expanding the determinant by the first row)
shows that*

$$\det(B_\gamma(n_0, \cdots, n_d))$$

$$= (-1)^d n_0 + (-1)^{d+1}[\pi - \sum_{k=2}^{d} \pi^{2k}]n_1 + (-1)^{d+1} \sum_{k=2}^{d} \pi^k n_k$$

$$= (-1)^d n_0 + (-1)^{d+1} \sum_{k=1}^{d} \pi^k n_k + (-1)^d \sum_{k=2}^{d} \pi^{2k} n_1,$$

*which does not vanish if $(n_0, \cdots, n_d) \in \mathbb{Z}^{d+1} \setminus \{(0, \cdots, 0)\}$, since π is a
transcendental number. This solves the case $s = d+1$. The general case
follows as a direct consequence of this one, since the matrices*

$$\begin{bmatrix} \pi & 1 & \pi^2 & \cdots & \pi^d & \pi^{d+1} & \cdots & \pi^s \\ \pi^2 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \ddots & \vdots \\ \pi^d & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ n_0 & n_1 & n_2 & \cdots & n_d & n_{d+1} & \cdots & n_s \end{bmatrix}$$

*contain $B_\gamma(n_0, \cdots, n_d)$ as a submatrix and, hence, have maximal rank for all
$s > d$ and all $(n_0, \cdots, n_s) \in \mathbb{Z}^{s+1} \setminus \{(0, \cdots, 0)\}$.*

Several distributional techniques have been used in the study of functional equations. In particular, Fourier transform of tempered distributions is used in [8] to introduce a method of solving some special functional equations. This motivates study Montel's type theorems in the distributional setting. Let $\mathcal{S}'(\mathbb{R}^d)$ denote the space of complex tempered distributions defined on $\mathbb{R}^d$.

Obviously, $\mathcal{S}'(\mathbb{R}^d)$ is a vector space and the operators Δ_h^s can be defined as endomorphisms of $\mathcal{S}'(\mathbb{R}^d)$ by the formula,

$$\Delta_h^s f\{\varphi\} = (-1)^s f\{\Delta_{-h}^s \varphi\}$$

for $s = 1, 2, \cdots$. Furthermore, if $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d$ is any multi-index and we denote by $D^\alpha f = \dfrac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \dfrac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \dfrac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} f$ the α-th generalized derivative of f, then $\Delta_h^m D^\alpha = D^\alpha \Delta_h^m$ for all $m \in \mathbb{N}$, and if $D^\alpha f = 0$ for all multi indices α with $|\alpha| = n$, then there exists $P \in \Pi_{n-1,\text{tot}}^d$ such that $P = f$ almost everywhere. Of course, if f is taken to be continuous, then $f = P$ everywhere. Finally, the structure theorem for tempered distributions guarantees that if $f \in \mathcal{S}'(\mathbb{R}^d)$ then there exist a slowly growing continuous function F and a natural number $n \in \mathbb{N}$ such that $f = D^{(n,n,\cdots,n)}F$ [47, page 98]. Tempered distributions are also interesting because the Fourier transform $\mathcal{F}$ can be defined as an automorphism $\mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ just imposing $\mathcal{F}(f)\{\varphi\} = f\{\mathcal{F}(\varphi)\}$, and the new operator preserves the main properties of the classical Fourier transform (see, for example, [14, page 144], [38, page 192, Theorem 7.15]).

Lemma 2.5. *Assume that* $f \in \mathcal{S}'(\mathbb{R}^d)$ *and let* $\Gamma = \Delta_{e_1}^1 \Delta_{e_2}^1 \cdots \Delta_{e_d}^1$ *and* $n \in \mathbb{N} \setminus \{0\}$. *Then* $D^{(n,n,\cdots,n)}f = 0$ *implies* $\Gamma^n f = 0$.

Proof. $f \in \mathcal{S}'(\mathbb{R}^d)$ implies that

$$\mathcal{F}(D^\alpha f)(\xi) = (i\xi)^\alpha \mathcal{F}(f)(\xi)$$

for every multi-index α and

$$\mathcal{F}(\Delta_{h_0}f(x))(\xi) = \mathcal{F}(f(x+h_0) - f(x))(\xi) = (e^{i<h_0,\xi>} - 1)\mathcal{F}(f)(\xi)$$

for every step $h_0 \in \mathbb{R}^d$. Hence, if $\mathcal{D} = D^{(1,1,\cdots,1)}$ and $\mathcal{D}(f) = 0$, then

$$0 = \mathcal{F}(\mathcal{D}(f))(\xi) = (i\xi)^{(1,\cdots,1)}\mathcal{F}(f)(\xi) = (i)^d \xi_1 \xi_2 \cdots \xi_d \mathcal{F}(f)(\xi). \qquad (2.2)$$

In particular, the support of $\mathcal{F}(f)$ is a subset of $V = \bigcup_{k=1}^d H_k$, where $H_k = \{\xi : \xi_k = 0\}$ is an hyperplane of $\mathbb{R}^d$ for every k. On the other hand,

$$\mathcal{F}(\Gamma(f))(\xi) = \left(\prod_{k=1}^d (e^{i<e_k,\xi>} - 1)\right)\mathcal{F}(f)(\xi) = \left(\prod_{k=1}^d (e^{i\xi_k} - 1)\right)\mathcal{F}(f)(\xi)$$
(2.3)

and, evidently, if the support of $\mathcal{F}(f)$ is a subset of V, then

$$\prod_{k=1}^d (e^{i\xi_k} - 1)\mathcal{F}(f)(\xi) = 0$$

(since $\mathcal{F}(f)$ vanishes on all points ξ such that $\prod_{k=1}^d (e^{i\xi_k} - 1) \neq 0$) and $\mathcal{F}(\Gamma(f))(\xi) = 0$, which implies $\Gamma(f) = 0$. Thus, if $f \in \mathcal{S}'(\mathbb{R}^d)$ and $\mathcal{D}(f) = 0$, then $\Gamma(f) = 0$, which is the case $n = 1$ of the lemma. Assume the result holds for $n-1$ and let $f \in \mathcal{S}'(\mathbb{R}^d)$ be such that $\mathcal{D}^n f = 0$. Then

$$0 = \mathcal{D}^n f = \mathcal{D}(\mathcal{D}^{n-1}f),$$

so that

$$0 = \Gamma(\mathcal{D}^{n-1} f) = \mathcal{D}^{n-1}(\Gamma f)$$

and the induction hypothesis implies that $\Gamma^{n-1}(\Gamma f) = 0$, which is what we wanted to prove.   □

We are now able to prove the following result:

Corollary 2.6. *Let $f \in \mathcal{S}'(\mathbb{R}^d)$ and let $H = h_1\mathbb{Z} + \cdots + h_s\mathbb{Z}$ be a dense subgroup of $\mathbb{R}^d$. Assume that $\Delta_{h_k}^{m+1} f = 0$, $k = 1, \cdots, s$. Then there exists $f^* \in \Pi_{sm,tot}^d$ such that $f = f^*$ almost everywhere.*

Proof. Let $f \in \mathcal{S}'(\mathbb{R}^d)$ satisfy the hypotheses of this corollary and let us take $n \in \mathbb{N}$ and $F : \mathbb{R}^d \to \mathbb{C}$ a continuous slowly growing function such that $f = D^{(n,n,\cdots,n)} F$. Then, for $1 \le k \le s$, we have that

$$
\begin{aligned}
0 &= \Delta_{h_k}^{m+1} f \\
&= \Delta_{h_k}^{m+1} D^{(n,n,\cdots,n)} F = D^{(n,n,\cdots,n)}(\Delta_{h_k}^{m+1} F)
\end{aligned}
$$

Hence, Lemma 2.5 implies that

$$0 = \Gamma^n(\Delta_{h_k}^{m+1} F) = \Delta_{h_k}^{m+1}(\Gamma^n F), \text{ for all } 1 \le k \le s,$$

and since $\Gamma^n(F)$ is continuous, we can apply Corollary 2.3 to $\Gamma^n(F)$ to conclude that $\Gamma^n(F)$ is a polynomial. In particular, F is of class $C^{(\infty)}$ and $f = D^{(n,n,\cdots,n)} F$ in distributional sense, which implies that f is equal almost everywhere to a continuous function f^* and this function f^* satisfies $\Delta_{h_k}^{m+1} f^* = 0$, $k = 1, \cdots, s$ in the classical sense. Thus, if we apply Corollary 2.3 to f^* we conclude that $f^* \in \Pi_{sm,\text{tot}}^d$ and $f = f^*$ almost everywhere. This concludes the proof.

                                                                               □

Obviously, we can resume all results proved in this section with the statement of the following generalized version of Montel's Theorem:

Theorem 2.7. *We suppose that s is a positive integer, and either of the following possibilities holds:*

1. *G is a finitely generated Abelian group with generators $h_1, \ldots, h_s$, and $f : G \to \mathbb{C}$ is a function.*
2. *G is a topological Abelian group, in which the elements $h_1, \ldots, h_s$ generate a dense subgroup in G, and $f : G \to \mathbb{C}$ is a continuous function.*

If f satisfies

$$\Delta_{h_k}^{m+1} f = 0 \tag{2.4}$$

for $k = 1, 2, \ldots, s$, then f is a polynomial of total degree $\le sm$ on G. Furthermore, if $G = \mathbb{R}^d$, sm is the best possible. Finally, if $G = \mathbb{R}^d$, the elements $h_1, \ldots, h_s$ generate a dense subgroup of $\mathbb{R}^d$, and f is a complex valued tempered distribution on $\mathbb{R}^d$ which satisfies (2.4), then $f = p$ almost everywhere for some $p \in \Pi_{sm,tot}^d$.

Note that, with completely different techniques, similar results have been recently demonstrated by Almira [2], Almira-Abu Helaiel [4] and Almira-Székelyhidi [7].

3. Montel-Popoviciu theorem in several variables setting

In this section we prove a result of Popoviciu's type for functions defined on the Euclidean space $\mathbb{R}^d$ for $d > 1$. We begin by a technical lemma showing that every polynomial $P \in \Pi^s_{m,\max}$ can be decomposed as a special sum involving polynomials of the form $A_k(t_1 + \theta_1 t_s, t_2 + \theta_2 t_s, \cdots, t_{s-1} + \theta_{s-1}t_s)$, with $A_k \in \Pi^{s-1}_{(s-1)m,\max}$ and $k = 0, \cdots, sm$.

Lemma 3.1. *Let $\{\theta_1, \cdots, \theta_{s-1}\} \subset \mathbb{R}\backslash\{0\}$. Then every polynomial $P \in \Pi^s_{m,\max}$ can be decomposed as a sum of the form*

$$P(t_1, \cdots, t_s) = \sum_{k=0}^{sm} A_k(t_1 + \theta_1 t_s, t_2 + \theta_2 t_s, \cdots, t_{s-1} + \theta_{s-1}t_s)t_s^k$$

where $A_k \in \Pi^{s-1}_{(s-1)m,\max}$ for $k = 0, \cdots, sm$.

Proof. Let $P(t_1, \cdots, t_s) = \sum_{i_1=0}^m \sum_{i_2=0}^m \cdots \sum_{i_s=0}^m a_{i_1,i_2,\cdots,i_s} t_1^{i_1} \cdots t_s^{i_s}$ and let us consider the change of variables given by $f_1 = t_1 + \theta_1 t_s, \cdots, f_{s-1} = t_{s-1} + \theta_{s-1}t_s$ and $f_s = t_s$. Then $t_k = f_k - \theta_k f_s$ for all $1 \leq k \leq s-1$, and $f_s = t_s$, so that

$$
\begin{aligned}
P(t_1, \cdots, t_s) &= \sum_{i_1=0}^m \cdots \sum_{i_s=0}^m a_{i_1,\cdots,i_s}(f_1 - \theta_1 f_s)^{i_1} \cdots (f_{s-1} - \theta_{s-1}f_s)^{i_{s-1}}(f_s)^{i_s} \\
&= \sum_{k=0}^{sm} A_k(f_1, f_2, \cdots, f_{s-1})f_s^k \\
&= \sum_{k=0}^{sm} A_k(t_1 + \theta_1 t_s, t_2 + \theta_2 t_s, \cdots, t_{s-1} + \theta_{s-1}t_s)t_s^k,
\end{aligned}
$$

where $A_k(f_1, \cdots, f_{s-1})$ is a polynomial of $s-1$ variables with degree at most $(s-1)m$ in each one of them, for $k = 0, \cdots, sm$. $\qquad\square$

Theorem 3.2 (Polynomials of the form $A(t_1+\theta_1 t_s, t_2+\theta_2 t_s, \cdots, t_{s-1}+\theta_{s-1}t_s)$). *Let $\{\theta_1, \cdots, \theta_{s-1}\} \subset \mathbb{R} \setminus \{0\}$ and let $P(t_1, \cdots, t_s) \in \Pi^s_{m,max}$. Then*

$$P(t_1, \cdots, t_s) = A_0(t_1 + \theta_1 t_s, t_2 + \theta_2 t_s, \cdots, t_{s-1} + \theta_{s-1}t_s),$$

with $A_0 \in \Pi^{s-1}_{(s-1)m,\max}$ if and only if there exists $W \subseteq \mathbb{R}^{s-1}$, a correct interpolation set for $\Pi^{s-1}_{(s-1)m,\max}$, such that, for all $\alpha = (\alpha_1, \cdots, \alpha_{s-1}) \in W$, there exists a sequence of vectors $\{(u_{1,n}, \cdots, u_{s-1,n}, u_{s,n})\}_{n=1}^\infty$ satisfying the following three conditions:

(i) $u_{j,n} + \theta_j u_{s,n} \to \alpha_j$ when $n \to \infty$, for $1 \leq j \leq s-1$.
(ii) $|u_{s,n}| \to \infty$ for $n \to \infty$, and
(iii) $\{P(u_{1,n}, u_{2,n}, \cdots, u_{s-1,n}, u_{s,n})\}_{n=1}^\infty$ is bounded.

To prove Theorem 3.2 we need first to state some technical results:

Lemma 3.3. *Let* $p(z) = a_0 + a_1 z + \cdots + a_N z^N \in \mathbb{C}[z]$ *be an ordinary polynomial of degree* N *(i.e.,* $a_N \neq 0$*) and let* $\xi \in \mathbb{C}$ *be a zero of* p. *Then*

$$|\xi| \leq max\{1, \sum_{k=0}^{N-1} \frac{|a_k|}{|a_N|}\}.$$

Proof. This is a well known fact, but we include the proof for the sake of completeness. If $|\xi| \leq 1$ we are done. Thus, let us assume $|\xi| > 1$. Obviously, $q(z) = \frac{1}{|a_N|} p(z) = \sum_{k=0}^{N-1} \frac{a_k}{a_N} z^k + z^N$ satisfies $q(\xi) = 0$. Hence

$$|\xi|^N = \left| \sum_{k=0}^{N-1} \frac{a_k}{a_N} \xi^k \right| \leq \sum_{k=0}^{N-1} \frac{|a_k|}{|a_N|} \max\{1, |\xi|, \cdots, |\xi|^{N-1}\} = \left(\sum_{k=0}^{N-1} \frac{|a_k|}{|a_N|} \right) |\xi|^{N-1}.$$

It follows that, in this case, $|\xi| \leq \sum_{k=0}^{N-1} \frac{|a_k|}{|a_N|}$, which is what we wanted to prove. $\square$

Lemma 3.4. *Let* $p(z) = a_0 + a_1 z + \cdots + a_N z^N \in \mathbb{C}[z]$ *be an ordinary polynomial of degree* N *(i.e.,* $a_N \neq 0$*) and assume that* $N \geq 1$. *Let* $\{q_n(z)\}_{n=1}^{\infty}$ *be a sequence of ordinary polynomials of degree* $\leq N$,

$$q_n(z) = a_{0n} + a_{1n} z + \cdots + a_{Nn} z^N,$$

and assume that

$$max\{|a_k - a_{kn}| : k = 0, 1, \cdots, N\} < |a_N|/2, \quad n = 1, 2, \cdots, \infty.$$

If $|w_n| \to +\infty$, *then* $|q_n(w_n)| \to \infty$.

Proof. Let $n \in \mathbb{N}$ and let ξ be a zero of $q_n(z)$. Then

$$|\xi| \leq \max\{1, \sum_{k=0}^{N-1} \frac{|a_{kn}|}{|a_{Nn}|}\},$$

and, since $|a_{kn}| \leq |a_{kn} - a_k| + |a_k| \leq \frac{|a_N|}{2} + |a_k|$, $|a_{Nn}| \geq \frac{|a_N|}{2}$, we conclude that

$$|\xi| \leq \max\{1, \sum_{k=0}^{N-1} \frac{2(\frac{|a_N|}{2} + |a_k|)}{|a_N|}\} =: M.$$

Thus, all zeroes of $q_n(z)$ belong to $B_M = \{z \in \mathbb{C} : |z| \leq M\}$ (for all n).

If $|w_n| \to \infty$, then $\text{dist}(w_n, B_M) \to \infty$.

On the other hand, if $\{\alpha_{kn}\}_{k=1}^{N}$ denotes the set of zeroes of $q_n(z)$, then

$$q_n(z) = a_{Nn} \prod_{k=1}^{N} (z - \alpha_{kn}),$$

so that

$$|q_n(w_n)| = |a_{Nn}| \prod_{k=1}^{N} |w_n - \alpha_{kn}| \geq \frac{|a_N|}{2} (\text{dist}(w_n, B_M))^N \to \infty. \quad (n \to \infty).$$

$\square$

Proof of Theorem 3.2. The necessity is obvious, since every polynomial of the form

$$P(t_1, \cdots, t_s) = A_0(t_1 + \theta_1 t_s, t_2 + \theta_2 t_s, \cdots, t_{s-1} + \theta_{s-1} t_s),$$

with $A_0 \in \Pi^{s-1}_{(s-1)m,\max}$, is uniformly bounded on strips of the form

$$\Gamma_{a,b} = \{(t_1, \cdots, t_s) : a_k \leq t_k + \theta_k t_s \leq b_k, \ k = 1, \cdots, s-1\},$$

where $\{a = (a_1, \cdots, a_{s-1}), b = (b_1, \cdots, b_{s-1})\} \subset \mathbb{R}^{s-1}$.

Let us now prove the sufficiency. Let $P(t_1, \cdots, t_s) \in \Pi^s_{m,\max}$. Then, for a certain $N \leq sm$, P admits a representation of the form

$$P(t_1, \cdots, t_s) = \sum_{k=0}^{N} A_k(t_1 + \theta_1 t_s, t_2 + \theta_2 t_s, \cdots, t_{s-1} + \theta_{s-1} t_s) t_s^k,$$

where A_k is a polynomial of $s-1$ variables with degree at most $(s-1)m$ in each one of them, for $0 \leq k \leq N$, and $A_N \neq 0$. We must prove $N = 0$.

Assume, on the contrary, that $N > 0$.

By hypothesis, $A_N(\alpha) \neq 0$ for a certain $\alpha = (\alpha_1, \cdots, \alpha_{s-1}) \in W$, since $A_N \in \Pi^{s-1}_{(s-1)m,max} \setminus \{0\}$ and W is a correct interpolation set for $\Pi^{s-1}_{(s-1)m,max}$. Consider the polynomial

$$p(z) = \sum_{i=0}^{N} A_i(\alpha) z^i$$

Now, the functions $A_i(t_1, \cdots, t_{s-1})$ are continuous and $\{u_{j,n} + \theta_j u_{s,n}\} \to \alpha_j$ for $n \to \infty$ and $1 \leq j \leq s-1$, so that

$$\{A_i(u_{1,n} + \theta_1 u_{s,n}, u_{2,n} + \theta_2 u_{s,n}, \cdots, u_{s-1,n} + \theta_{s-1} u_{s,n})\} \to A_i(\alpha)$$

for $i \in \{0, 1, \cdots, N\}$. Thus, we can assume with no loss of generality, that $|A_i(u_{1,n} + \theta_1 u_{s,n}, u_{2,n} + \theta_2 u_{s,n}, \cdots, u_{s-1,n} + \theta_{s-1} u_{s,n}) - A_i(\alpha)| < |A_N(\alpha)|/2$, $i = 0, 1, \cdots, N, n \in \mathbb{N}$.

Hence, if we consider the sequence of polynomials

$$q_n(z) = \sum_{i=0}^{N} A_i(u_{1,n} + \theta_1 u_{s,n}, u_{2,n} + \theta_2 u_{s,n}, \cdots, u_{s-1,n} + \theta_{s-1} u_{s,n}) z^i,$$

$n = 1, 2, \cdots$, then Lemma 3.4 implies that $|q_n(u_{s,n})| \to \infty$ for $n \to \infty$. But $|q_n(u_{s,n})| = |P(u_{1,n}, u_{2,n}, \cdots, u_{s-1,n}, u_{s,n})|$ is bounded. Hence, if $N > 0$ we get a contradiction. It follows that $N = 0$ and

$$P(t_1, \cdots, t_s) = A_0(t_1 + \theta_1 t_s, t_2 + \theta_2 t_s, \cdots, t_{s-1} + \theta_{s-1} t_s),$$

with $A_0 \in \Pi^{s-1}_{(s-1)m,\max}$. $\qquad\square$

In the following $\Delta^{m+1}_{(\theta_1, \cdots, \theta_d)} f(x)$ denotes the usual

$$\Delta^{m+1}_{(\theta_1, \cdots, \theta_d)} f(x) = \sum_{k=0}^{m+1} \binom{m+1}{k} (-1)^{m+1-k} f(x + k(\theta_1, \cdots, \theta_d)).$$

Corollary 3.5. *Assume that $f : \mathbb{R}^d \to \mathbb{R}$ is bounded on a certain open set $U \subseteq \mathbb{R}^d$, $U \neq \emptyset$, and $H = \mathbb{Z}^d + (\theta_1, \theta_2, \cdots, \theta_d)\mathbb{Z}$ is a dense subgroup of $\mathbb{R}^d$. If $\beta = \{e_k\}_{k=1}^d$ denotes the canonical basis of $\mathbb{R}^d$ and f satisfies*

$$\Delta_{e_k}^{m+1} f(x) = 0, \ for \ k = 1, \cdots, d; \quad \Delta_{(\theta_1,\cdots,\theta_d)}^{m+1} f(x) = 0,$$

and $P_{a,\gamma}$ denotes the polynomial constructed in Lemma 2.1, for $\gamma = \beta \cup \{(\theta_1, \cdots, \theta_d)\}$, then

$$P_{a,\gamma}(t_1, \cdots, t_{d+1}) = A(t_1 + \theta_1 t_{d+1}, t_2 + \theta_2 t_{d+1}, \cdots, t_d + \theta_d t_{d+1}),$$

with $A \in \Pi_{dm,\max}^d$.

Proof. There is no loss of generality if we assume that $a = 0$. Indeed, if we use the notation $P_{a,\gamma}(f)$ for the polynomial constructed in Lemma 2.1 for the function f, and we take $g(x) = f(x + a)$, then it is clear that $P_{a,\gamma}(f) = P_{0,\gamma}(g)$.

Let $W \subseteq U$ be a correct interpolation set for $\Pi_{dm,\max}^d$ whose entries have rational coordinates (such set obviously exists since U is open). Then, for every $\alpha = (\alpha_1, \cdots, \alpha_d) \in W$ there exists a sequence $\{(i_{1,n}, \cdots, i_{d,n}) + i_{d+1,n}(\theta_1, \cdots, \theta_d)\}_{n=1}^\infty$ which is contained in U and satisfies

$$\lim_{n \to \infty} (i_{1,n}, \cdots, i_{d,n}) + i_{d+1,n}(\theta_1, \cdots, \theta_d) = \alpha.$$

The density of $H = \mathbb{Z}^d + (\theta_1, \theta_2, \cdots, \theta_d)\mathbb{Z}$ in $\mathbb{R}^d$ implies that the numbers $\{1, \theta_1, \cdots, \theta_d\}$ form a linearly independent system over $\mathbb{Q}$. In particular, θ_k is an irrational number for $k = 1, \cdots, d$. From this, and from the convergence of $i_{k,n} + \theta_k i_{d+1,n}$ to $\alpha_k \in \mathbb{Q}$, it follows that $|i_{d+1,n}| \to \infty$ for $n \to \infty$. On the other hand,

$$P_{0,\gamma}(i_{1,n}, \cdots, i_{d,n}, i_{d+1,n}) = f((i_{1,n}, \cdots, i_{d,n}) + i_{d+1,n}(\theta_1, \cdots, \theta_d))$$

is bounded. Thus, we can apply Theorem 3.2 to $P_{0,\gamma}$, completing the proof.
□

Now we are ready to prove the main result of this section:

Theorem 3.6 (Montel-Popoviciu theorem for several variables). *Assume that $H = \mathbb{Z}^d + (\theta_1, \theta_2, \cdots, \theta_d)\mathbb{Z}$ is a dense subgroup of $\mathbb{R}^d$. If $\beta = \{e_k\}_{k=1}^d$ denotes the canonical basis of $\mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}$ satisfies*

$$\Delta_{e_k}^{m+1} f(x) = 0, \ for \ k = 1, \cdots, d, \ and \ \Delta_{(\theta_1,\cdots,\theta_d)}^{m+1} f(x) = 0,$$

and is continuous at every point of a set $W \subseteq \mathbb{R}^d$ which is a correct interpolation set for $\Pi_{dm,\max}^d$, then $f \in \Pi_{m,\max}^d$.

Proof. We divide the proof into two parts. In the first one we prove that f is an ordinary polynomial which belongs to $\Pi_{dm,\max}^d$. In the second part we improve the result by showing that $f \in \Pi_{m,\max}^d$, which is a smaller space of polynomials.

Part I: To show that $f \in \Pi_{dm,\max}^d$, note that continuity of f at just one point implies that f is bounded on a certain nonempty open set $U \subseteq \mathbb{R}^d$, so that we

can apply Corollary 3.5 to f. In particular, for every $a = (a_1, \cdots, a_d) \in \mathbb{R}^d$, there exists $A_a \in \Pi_{dm,\max}^d$ satisfying

$$A_a((n_1, \cdots, n_d) + n_{d+1}(\theta_1, \cdots, \theta_d)) = f(a + (n_1, \cdots, n_d) + n_{d+1}(\theta_1, \cdots, \theta_d))$$

for all $(n_1, \cdots, n_{d+1}) \in \mathbb{Z}^{d+1}$. The result follows if we prove that, for any $a \in \mathbb{R}^d$, the relation

$$A_a(x) = A_{\mathbf{0}}(x - a) \tag{3.1}$$

holds for all $x \in \mathbb{R}^d$ (here, $\mathbf{0}$ denotes the zero vector $(0, \cdots, 0) \in \mathbb{R}^d$). To prove this, it is enough to take into account that, if (3.1) holds true, then, for each $a \in \mathbb{R}^d$, we have that

$$f(a) = A_a(\mathbf{0}) = A_{\mathbf{0}}(-a).$$

In other words, $f(x) = A_{\mathbf{0}}(-x) \in \Pi_{dm,\max}^d$.

Let us demonstrate the validity of (3.1). We fix $a \in \mathbb{R}^d$ and we define the polynomial $C(x) = A_{\mathbf{0}}(x - a)$. Let us show that $C = A_a$. Obviously, $C \in \Pi_{dm,\max}^d$. Assume that f is continuous at every point of a set $W \subseteq \mathbb{R}^d$ which is a correct interpolation set for $\Pi_{dm,\max}^d$. Obviously, $C \in \Pi_{dm,\max}^d$, so that $C = A_a$ if and only if $C_{|W} = (A_a)_{|W}$. Take $\alpha \in W$. The density of H in $\mathbb{R}^d$ implies that, for certain sequences of vectors $(i_{1,n}, \cdots, i_{d+1,n}), (j_{1,n}, \cdots, j_{d+1,n}) \in \mathbb{Z}^{d+1}$, we will have that

$$\begin{aligned}
\alpha &= \lim_{n \to \infty} [(i_{1,n}, \cdots, i_{d,n}) + i_{d+1,n}(\theta_1, \cdots, \theta_d)] \\
&= \lim_{n \to \infty} [a + (j_{1,n}, \cdots, j_{d,n}) + j_{d+1,n}(\theta_1, \cdots, \theta_d)].
\end{aligned}$$

Hence, the continuity of f at α implies that

$$\begin{aligned}
f(\alpha) &= \lim_{n \to \infty} [f((i_{1,n}, \cdots, i_{d,n}) + i_{d+1,n}(\theta_1, \cdots, \theta_d))] \\
&= \lim_{n \to \infty} [A_{\mathbf{0}}((i_{1,n}, \cdots, i_{d,n}) + i_{d+1,n}(\theta_1, \cdots, \theta_d))] \\
&= A_{\mathbf{0}}(\alpha)
\end{aligned}$$

and

$$\begin{aligned}
f(\alpha) &= \lim_{n \to \infty} [f(a + (j_{1,n}, \cdots, j_{d,n}) + j_{d+1,n}(\theta_1, \cdots, \theta_d))] \\
&= \lim_{n \to \infty} [A_a(j_{1,n}, \cdots, j_{d,n}) + j_{d+1,n}(\theta_1, \cdots, \theta_d))] \\
&= A_a(\alpha - a) = C(\alpha)
\end{aligned}$$

It follows that $C(\alpha) = A_{\mathbf{0}}(\alpha)$ for all $\alpha \in W$. This concludes the proof that $f \in \Pi_{dm,\max}^d$.

Part II: Let us prove $f \in \Pi_{m,\max}^d$. By Part I we know that $f \in \Pi_{dm,\max}^d$. Thus, if we apply the set of equations

$$\Delta_{e_k}^{m+1} f(x) = 0, \text{ for } k = 1, \cdots, d.$$

to $f \in \Pi^d_{dm,\max}$, we get that $f \in \Pi^d_{m,\max}$. Indeed, if $f \in \Pi^d_{N,\max}$, then for every $k \in \{1, \cdots, d\}$ we can uniquely decompose f as a sum

$$f = \sum_{i=1}^{N} \phi_i(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_d)x_k^i,$$

with $\phi_1, \cdots, \phi_N$ polynomials in $d-1$ variables and $\phi_N \neq 0$. We must show that $N \leq m$. Now,

$$
\begin{aligned}
\Delta_{e_k}^{m+1} f(x) &= \sum_{i=0}^{N} \phi_i(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_d)\Delta_1^{m+1} x_k^i \\
&= \sum_{i=m+1}^{N} \phi_i(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_d)\Delta_1^{m+1} x_k^i,
\end{aligned}
$$

which is equal to zero if and only if $N \leq m$. This ends the proof for both parts. $\qquad \square$

Corollary 3.7. *Let $H = h_1\mathbb{Z} + \cdots + h_s\mathbb{Z}$ be a finitely generated subgroup of $\mathbb{R}^d$ and assume that $\mathbb{Z}^d + (\theta_1, \theta_2, \cdots, \theta_d)\mathbb{Z} \subseteq H$ for certain system of real numbers $\{\theta_k\}_{k=1}^d$ such that $\{1, \theta_1, \cdots, \theta_d\}$ is $\mathbb{Q}$-linearly independent. If $f : \mathbb{R}^d \to \mathbb{R}$ satisfies*

$$\Delta_{h_k}^{m+1} f(x) = 0, \text{ for } k = 1, \cdots, s$$

and is continuous at every point of a set $W \subseteq \mathbb{R}^d$ which is a correct interpolation set for $\Pi^d_{dsm,\max}$, then $f \in \Pi^d_{sm,\max}$.

Proof. Applying Corollary 2.3 to f we conclude that $\Delta_h^{sm+1} f = 0$ for all $h \in H$. In particular, we can use Theorem 3.6 with this function just substituting m by sm. $\qquad \square$

4. Optimality

In this section we prove that Popoviciu's original theorem is optimal, and we consider the optimality of Theorem 3.6 in the several variables setting. Let us start with the case $d = 1$. Consider the function

$$f(x) = \begin{cases} x(x-1)\cdots(x-(m-1)) & x \in h_1\mathbb{Z} + h_2\mathbb{Z} \\ 0 & \text{otherwise} \end{cases},$$

where $H = h_1\mathbb{Z} + h_2\mathbb{Z}$ is assumed to be dense in $\mathbb{R}$. If $x \in H$, then $\{x + kh_i\}_{k=0}^{m+1} \subseteq H$ for $i = 1, 2$, so that $f_{|\{x+kh_i\}_{k=0}^{m+1}} = p_{|\{x+kh_i\}_{k=0}^{m+1}}$, where $p(x) = x(x-1)\cdots(x-(m-1))$. Thus, for $i \in \{1, 2\}$,

$$
\begin{aligned}
\Delta_{h_i}^{m+1} f(x) &= \sum_{k=0}^{m+1} \binom{m+1}{k}(-1)^{m+1-k} f(x + kh_i) \\
&= \sum_{k=0}^{m+1} \binom{m+1}{k}(-1)^{m+1-k} p(x + kh_i) \\
&= \Delta_{h_i}^{m+1} p(x) = 0,
\end{aligned}
$$

since $p \in \Pi_m$. On the other hand, if $x \notin H$, then $\{x + kh_i\}_{k=0}^{m+1} \cap H = \emptyset$, so that $f_{|\{x+kh_i\}_{k=0}^{m+1}} = 0$, and $\Delta_{h_i}^{m+1} f(x) = 0$. This proves that $\Delta_{h_i}^{m+1} f(x) = 0$ for all x, for $i = 1, 2$. Furthermore, f is continuous at m points and it is not an ordinary polynomial.

Let us now consider the case $d > 1$.

Let $H = \mathbb{Z}^d + (\theta_1, \cdots, \theta_d)\mathbb{Z}$ be a dense subgroup of $\mathbb{R}^d$, and let us consider the function
$F(x_1, \cdots, x_d) = F_1(x_1, \cdots, x_d) + \cdots + F_d(x_1, \cdots, x_d)$, where $F_i(x_1, \cdots, x_d) = g(x_i)$ and

$$g(x) = \begin{cases} x(x-1)\cdots(x-(m-1)) & x \in \theta_1\mathbb{Z} + \theta_2\mathbb{Z} + \cdots + \theta_d\mathbb{Z} \\ 0 & \text{otherwise} \end{cases}.$$

Let us compute $\Delta_{e_k}^{m+1} F$ and $\Delta_{(\theta_1, \cdots, \theta_d)}^{m+1} F$. First of all, it is easy to check that $\Delta_{e_k}^{m+1} F_i = 0$ for $1 \leq i, k \leq d$ and, hence, $\Delta_{e_k}^{m+1} F = \sum_{i=1}^d \Delta_{e_k}^{m+1} F_i = 0$, for $k = 1, \cdots, d$. On the other hand, if $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, then

$$\begin{aligned}
\Delta_{(\theta_1, \cdots, \theta_d)}^{m+1} F(x) &= \sum_{k=0}^{m+1} \binom{m+1}{k} (-1)^{m+1-k} F(x + k(\theta_1, \cdots, \theta_d)) \\
&= \sum_{k=0}^{m+1} \binom{m+1}{k} (-1)^{m+1-k} F(x_1 + k\theta_1, \cdots, x_d + k\theta_d) \\
&= \sum_{k=0}^{m+1} \binom{m+1}{k} (-1)^{m+1-k} \sum_{i=1}^d g(x_i + k\theta_i) \\
&= \sum_{i=1}^d \left(\sum_{k=0}^{m+1} \binom{m+1}{k} (-1)^{m+1-k} g(x_i + k\theta_i) \right) \\
&= \sum_{i=1}^d \Delta_{\theta_i}^{m+1} g(x_i).
\end{aligned}$$

Now, if $t \in \Gamma = \theta_1\mathbb{Z} + \cdots + \theta_d\mathbb{Z}$, then $\{t + k\theta_i\}_{k=0}^{m+1} \subseteq \Gamma$, so that $g_{|\{t+k\theta_i\}_{k=0}^{m+1}} = p_{|\{t+k\theta_i\}_{k=0}^{m+1}}$, where $p(t) = t(t-1)\cdots(t-(m-1))$, and $\Delta_{\theta_i}^{m+1} g(t) = 0$. If $t \notin \Gamma$, then $\{t + k\theta_i\}_{k=0}^{m+1} \cap \Gamma = \emptyset$ and $\Delta_{\theta_i}^{m+1} g(t) = 0$. This means that, for all $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, $\Delta_{\theta_i}^{m+1} g(x_i) = 0$. Hence $\Delta_{(\theta_1, \cdots, \theta_d)}^{m+1} F = 0$.

Now F is continuous at $W = \{(i_1, \cdots, i_d) : 0 \leq i_k \leq m - 1, \text{ for all } 1 \leq k \leq d\}$, which is a correct interpolation set for $\Pi_{m-1,\max}^d$. If we take into account that the conclusion of Theorem 3.6 is that $f \in \Pi_{m,\max}^d$, it seems natural to claim that this example shows that Theorem 3.6 is optimal (or near optimal) also for $d > 1$.

References

1. J. ACZEL, J. D'HOMBRES, *Functional equations in several variables,* Encyclopedia of Maths. and its Appl. **31**, Cambridge University Press, 1989.

2. J. M. Almira,   Montel's theorem and subspaces of distributions which are Δ^m-invariant, *Numer. Functional Anal. Optimiz.* **35** (4) (2014) 389-403.

3. J. M. Almira, A. J. López-Moreno,   On solutions of the Fréchet functional equation, *J. Math. Anal. Appl.* **332** (2007), 1119–1133.

4. J. M. Almira, K. F. Abu-Helaiel,   On Montel's theorem in several variables, to appear in *Carpathian Journal of Mathematics,* 2014, available at arXiv:1310.3378.

5. J. M. Almira, K. F. Abu-Helaiel,   A note on invariant subspaces and the solution of certain classical functional equations, *Annals of the Tiberiu Popoviciu Seminar on Functional Equations, Approximation and Convexity,* **11** (2013) 3-17.

6. J. M. Almira, K. F. Abu-Helaiel,   A qualitative description of graphs of discontinuous polynomials, to appear in *Annals of Functional Analysis,* 2015; available at arXiv:1401.3273.

7. J. M. Almira, L. Székelyhidi, Local polynomials and the Montel Theorem, manuscript, submitted, 2014.

8. J. A. Baker Functional equations, tempered distributions and Fourier transforms, *Trans. Amer. Math. Soc.* **315** (1989), no.1, 57-68.

9. S. Banach,   Sur l'equation fontionnelle $f(x + y) = f(x) + f(y)$, *Fundamenta Mathematicae* **1** (1920) 123-124.

10. K. Baron, W. Jarczyk,   Recent results on functional equations in a single variable, perspectives and open problems, *Aequationes Mathematicae* **61** (2001) 1-48.

11. Z. Ciesielski,   Some properties of convex functions of higher order, *Ann. Polon. Math.* **7** (1959) 1-7.

12. G. Darboux,   Memoire sur les fonctions discontinues, *Ann. Sci. École Norm. Sup.* **4** (1875) 57-112.

13. D. Z. Djoković,   A representation theorem for $(X_1 - 1)(X_2 - 1) \cdots (X_n - 1)$ and its applications, *Ann. Polon. Math.* **22** (1969/1970) 189-198.

14. W. F. Donoghue, Jr.,   *Distributions and Fourier Transforms,* Academic Press, New York and London, 1969.

15. M. Fréchet,   Une definition fonctionelle des polynomes, *Nouv. Ann.* **9** (1909), 145-162.

16. R. Ger, On some properties of polynomial functions, *Ann. Pol. Math.* **25** (1971) 195-203.

17. R. Ger,   On extensions of polynomial functions, *Results in Mathematics* **26** (1994), 281-289.

18. G. Hamel, Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung $f(x + y) = f(x) + f(y)$, *Math. Ann.* **60** (1905) 459-462.

19. E. Isaacson, H. B. Keller, *Analysis of Numerical Methods,* Wiley, New York, 1966.

20. D. H. Hyers, G. Isac, T. M. Rassias,   *Stability of functional equations in several variables,* Birkhäuser, 1998.

21. C. G. J. Jacobi,   De usu theoriae integralium ellipticorum et integralium abelianorum in analysi diophantea, Werke **2** (1834) 53-55.

22. A. JÁRAI, *Regularity properties of functional equations in several variables,* Springer Verlag, 2005.

23. A. JÁRAI, L. SZÉKELYHIDI, Regularization and General Methods in the Theory of Functional Equations, *Aequationes Mathematicae* **52** (1996) 10-29.

24. G. A. JONES, D. SINGERMAN, *Complex functions. An algebraic and geometric viewpoint*, Cambridge Univ. Press, 1987.

25. M. KORMES, On the functional equation $f(x + y) = f(x) + f(y)$, *Bulletin of the Amer. Math. Soc.* **32** (1926) 689-693.

26. M. KUCZMA, *An introduction to the theory of functional equations and inequalities,* (Second Edition, Edited by A. Gilányi), Birkhäuser, 2009.

27. M. KUCZMA, On some analogies between measure and category and their applications in the theory of additive functions, *Ann. Math. Sil.*No.**13** (1985), 155-162.

28. M. KUCZMA, On measurable functions with vanishing differences, *Ann. Math. Sil.* **6** (1992) 42-60.

29. S. KUREPA, A property of a set of positive measure and its application, *J. Math. Soc. Japan* **13** (1) (1961) 13-19.

30. M. LACZKOVICH, Polynomial mappings on Abelian groups, *Aequationes Mathematicae* **68** (3) (2004) 177-199.

31. M. A. MCKIERNAN, On vanishing n-th ordered differences and Hamel bases, *Ann. Pol. Math.* **19** (1967) 331-336.

32. P. MONTEL, Sur un théoreme du Jacobi, *Comptes Rend. Acad. Sci. París,* **201** (1935) 586.

33. P. MONTEL, Sur quelques extensions d'un théorème de Jacobi, *Prace Matematyczno-Fizyczne* **44** (1) (1937) 315-329.

34. ZS. PÁLES, Problems in the regularity theory of functional equations, *Aequationes Mathematicae* **63** (2002) 1-17.

35. T. POPOVICIU, Sur quelques propits des fonctions d'une ou de deux variables relles, Thse, Paris, 12 June 1933. Published in *Mathematica* vol. VIII, 1934, pp. 1-85.

36. T. POPOVICIU, Remarques sur la dfinition fonctionnelle d'un polynme d'une variable relle, *Mathematica (Cluj)* **12** (1936) 5-12.

37. TH. RASSIAS, J. BRZDEK, *Functional Equations in Mathematical Analysis,* Springer Verlag , 2011.

38. W. RUDIN, *Functional Analysis* (Second Edition), McGraw-Hill, 1991.

39. R. SAN JUAN, Una aplicación de las aproximaciones diofánticas a la ecuación funcional $f(x_1 + x_2) = f(x_1) + f(x_2)$, *Publicaciones del Inst. Matemático de la Universidad Nacional del Litoral* **6** (1946) 221-224.

40. W. SIERPINSKY, Sur l'equation fontionnelle $f(x+y) = f(x)+f(y)$, *Fundamenta Mathematicae* **1** (1920) 116-122.

41. H. STEINHAUS, Sur les distances des points dans les ensembles de mesure positive, *Fundamenta Mathematicae* **1** (1920) 93-104.

42. L. SZÉKELYHIDI, *Convolution type functional equations on topological abelian groups,* World Scientific, 1991.

43. L. SZÉKELYHIDI, *Discrete spectral synthesis and its applications,* Springer, 2006.

44. L. Székelyhidi, *Harmonic and Spectral Analysis,* World Scientific, 2014.

45. L. Székelyhidi, On Frchets functional equation, to appear in *Monatshefte für Mathematik,* 2014.

46. V. Vitrih, *Correct interpolation problems in multivariate interpolation spaces,* Ph. Thesis, Department of Mathematics, University of Ljubljana, 2010.

47. V. S. Vladimirov, *Generalized functions in Mathematical Analysis,* Mir Publishers, Moscow, 1979.

48. M. Waldschmidt, *Topologie des Points Rationnels,* Cours de Troisième Cycle 1994/95 Université P. et M. Curie (Paris VI), 1995.

A. G. Aksoy
Department of Mathematics. Claremont McKenna College,
Claremont, CA, 91711, USA.
e-mail: `aaksoy@cmc.edu`

J. M. Almira
Departamento de Matemáticas. Universidad de Jaén,
E.P.S. Linares, C/ Alfonso X el Sabio, 28,
23700 Linares (Jaén) Spain.
e-mail: `jmalmira@ujaen.es`